Linear algebra review

- vector space, subspaces
- independence, basis, dimension
- nullspace and range
- left and right invertibility

Vector spaces

a *vector space* or *linear space* (over the reals) consists of

- a set $\mathcal{V}$
- a vector sum $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$
- a scalar multiplication $: \mathbb{R} \times \mathcal{V} \to \mathcal{V}$
- a distinguished element $0 \in \mathcal{V}$

which satisfy a list of properties

Vector space axioms

- $x + y = y + x$, $\forall x, y \in \mathcal{V}$ — $+$ *is commutative*
- $(x + y) + z = x + (y + z)$, $\quad \forall x, y, z \in \mathcal{V}$ — $+$ *is associative*
- $0 + x = x$, $\forall x \in \mathcal{V}$ — 0 *is additive identity*
- $\forall x \in \mathcal{V} \;\; \exists (-x) \in \mathcal{V}$ s.t. $x + (-x) = 0$ — *existence of additive inverse*
- $(\alpha\beta)x = \alpha(\beta x)$, $\quad \forall \alpha, \beta \in \mathbb{R} \quad \forall x \in \mathcal{V}$ — *scalar mult. is associative*
- $\alpha(x + y) = \alpha x + \alpha y$, $\quad \forall \alpha \in \mathbb{R} \;\; \forall x, y \in \mathcal{V}$ — *right distributive rule*
- $(\alpha + \beta)x = \alpha x + \beta x$, $\quad \forall \alpha, \beta \in \mathbb{R} \;\; \forall x \in \mathcal{V}$ — *left distributive rule*
- $1x = x$, $\quad \forall x \in \mathcal{V}$ — 1 *is multiplicative identity*

Examples

- $\mathcal{V}_1 = \mathbb{R}^n$, with standard (componentwise) vector addition and scalar multiplication
- $\mathcal{V}_2 = \{0\}$ (where $0 \in \mathbb{R}^n$)
- $\mathcal{V}_3 = \textbf{span}(v_1, v_2, \ldots, v_k)$ where

  $$\textbf{span}(v_1, v_2, \ldots, v_k) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_i \in \mathbb{R}\}$$

  and $v_1, \ldots, v_k \in \mathbb{R}^n$

Subspaces

- a *subspace* of a vector space is a *subset* of a vector space which is itself a vector space
- roughly speaking, a subspace is closed under vector addition and scalar multiplication
- examples $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ above are subspaces of $\mathbb{R}^n$

Vector spaces of functions

- $\mathcal{V}_4 = \{x : \mathbb{R}_+ \to \mathbb{R}^n \mid x \text{ is differentiable}\}$, where vector sum is sum of functions:

$$(x + z)(t) = x(t) + z(t)$$

  and scalar multiplication is defined by

$$(\alpha x)(t) = \alpha x(t)$$

  (a *point* in $\mathcal{V}_4$ is a *trajectory* in $\mathbb{R}^n$)

- $\mathcal{V}_5 = \{x \in \mathcal{V}_4 \mid \dot{x} = Ax\}$
  (*points* in $\mathcal{V}_5$ are *trajectories* of the linear system $\dot{x} = Ax$)

- $\mathcal{V}_5$ is a subspace of $\mathcal{V}_4$

(Euclidean) norm

for $x \in \mathbb{R}^n$ we define the (Euclidean) norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^\mathsf{T} x}$$

$\|x\|$ measures length of vector (from origin)

important properties:

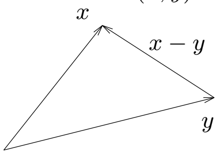
- $\|\alpha x\| = |\alpha| \|x\|$ — *homogeneity*
- $\|x + y\| \leq \|x\| + \|y\|$ — *triangle inequality*
- $\|x\| \geq 0$ — *nonnegativity*
- $\|x\| = 0 \iff x = 0$ — *definiteness*

RMS value and (Euclidean) distance

root-mean-square (RMS) value of vector $x \in \mathbb{R}^n$:

$$\textbf{rms}(x) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^2 \right)^{1/2} = \frac{\|x\|}{\sqrt{n}}$$

norm defines distance between vectors: $\textbf{dist}(x, y) = \|x - y\|$



Independent set of vectors

a set of vectors $\{v_1, v_2, \ldots, v_k\}$ is *independent* if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0 \quad \Longrightarrow \quad \alpha_1 = \alpha_2 = \cdots = 0$$

some equivalent conditions:

- coefficients of $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$ are uniquely determined, *i.e.*,

  $$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$$

  implies $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k$
- no vector v_i can be expressed as a linear combination of the other vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$

Basis and dimension

set of vectors $\{v_1, v_2, \dots, v_k\}$ is called a *basis* for a vector space $\mathcal{V}$ if

$$\mathcal{V} = \textbf{span}(v_1, v_2, \dots, v_k)$$

and

$\{v_1, v_2, \dots, v_k\}$ is independent

- equivalently, every $v \in \mathcal{V}$ *can be uniquely* expressed as

$$v = \alpha_1 v_1 + \cdots + \alpha_k v_k$$

- for a given vector space $\mathcal{V}$, the number of vectors in any basis is the same
- number of vectors in any basis is called the *dimension* of $\mathcal{V}$, denoted $\textbf{dim}\,\mathcal{V}$

Nullspace of a matrix

the *nullspace* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\textbf{null}(A) = \{\, x \in \mathbb{R}^n \mid Ax = 0 \,\}$$

- $\textbf{null}(A)$ is set of vectors mapped to zero by $y = Ax$
- $\textbf{null}(A)$ is set of vectors orthogonal to all rows of A

$\textbf{null}(A)$ gives *ambiguity* in x given $y = Ax$:

- if $y = Ax$ and $z \in \textbf{null}(A)$, then $y = A(x + z)$
- conversely, if $y = Ax$ and $y = A\tilde{x}$, then $\tilde{x} = x + z$ for some $z \in \textbf{null}(A)$

$\textbf{null}(A)$ is also written $\mathcal{N}(A)$

Zero nullspace

A is called *one-to-one* if 0 is the only element of its nullspace

$$\boxed{\mathbf{null}(A) = \{0\}}$$

Equivalently,

- x can always be uniquely determined from $y = Ax$
  ($i.e.$, the linear transformation $y = Ax$ doesn't 'lose' information)
- mapping from x to Ax is one-to-one: different x's map to different y's
- columns of A are independent (hence, a basis for their span)
- A has a *left inverse*, $i.e.$, there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $BA = I$
- $A^{\mathsf{T}}A$ is invertible

Two interpretations of nullspace

suppose $z \in \textbf{null}(A)$, and $y = Ax$ represents *measurement* of x

- z is undetectable from sensors — get zero sensor readings
- x and $x + z$ are indistinguishable from sensors: $Ax = A(x + z)$

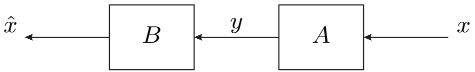
$\textbf{null}(A)$ characterizes *ambiguity* in x from measurement $y = Ax$

alternatively, if $y = Ax$ represents *output* resulting from input x

- z is an input with no result
- x and $x + z$ have same result

$\textbf{null}(A)$ characterizes *freedom of input choice* for given result

Left invertibility and estimation



- apply left-inverse B at output of A
- then estimate $\hat{x} = BAx = x$ as desired
- *non-unique:* both B and C are left inverses of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Range of a matrix

the *range* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\textbf{range}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

range(A) can be interpreted as

- the set of vectors that can be 'hit' by linear mapping $y = Ax$
- the span of columns of A
- the set of vectors y for which $Ax = y$ has a solution

range(A) is also written $\mathcal{R}(A)$

Onto matrices

A is called *onto* if **range**$(A) = \mathbb{R}^m$

equivalently,

- $Ax = y$ can be solved in x for any y
- columns of A span $\mathbb{R}^m$
- A has a *right inverse*, $i.e.$, there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $AB = I$
- rows of A are independent
- **null**$(A^\mathsf{T}) = \{0\}$
- AA^T is invertible

Interpretations of range

suppose $v \in \textbf{range}(A), w \notin \textbf{range}(A)$

$y = Ax$ represents *measurement* of x

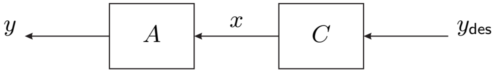
- $y = v$ is a *possible* or *consistent* sensor signal
- $y = w$ is *impossible* or *inconsistent*; sensors have failed or model is wrong

$y = Ax$ represents *output* resulting from input x

- v is a possible result or output
- w cannot be a result or output

range(A) characterizes the *possible results* or *achievable outputs*

Right invertibility and control



- apply right-inverse C at *input* of A
- then output $y = ACy_{\text{des}} = y_{\text{des}}$ as desired

Inverse

$A \in \mathbb{R}^{n \times n}$ is *invertible* or *nonsingular* if it has both a left and right inverse

equivalent conditions:

- columns of A are a basis for $\mathbb{R}^n$
- rows of A are a basis for $\mathbb{R}^n$
- $y = Ax$ has a unique solution x for every $y \in \mathbb{R}^n$
- $\textbf{null}(A) = \{0\}$
- $\textbf{range}(A) = \mathbb{R}^n$

Inverse

if a matrix A has both a left inverse and a right inverse, then they are equal

$$BA = I \text{ and } AC = I \qquad \Longrightarrow \qquad B = C$$

- hence if A is invertible then the inverse is unique
- $AA^{-1} = A^{-1}A = I$

Interpretations of inverse

suppose $A \in \mathbb{R}^{n \times n}$ has inverse $B = A^{-1}$

- mapping associated with B undoes mapping associated with A (applied either before or after!)
- $x = By$ is a perfect (pre- or post-) *equalizer* for the *channel* $y = Ax$
- $x = By$ is unique solution of $Ax = y$

Dual basis interpretation

- let a_i be columns of A, and $\tilde{b}_i^\mathsf{T}$ be rows of $B = A^{-1}$
- from $y = x_1 a_1 + \cdots + x_n a_n$ and $x_i = \tilde{b}_i^\mathsf{T} y$, we get

$$y = \sum_{i=1}^{n} (\tilde{b}_i^\mathsf{T} y) a_i$$

  thus, inner product with *rows of inverse matrix* gives the coefficients in the *expansion of a vector in the columns of the matrix*
- $\{\tilde{b}_1, \ldots, \tilde{b}_n\}$ and $\{a_1, \ldots, a_n\}$ are called *dual bases*

Change of coordinates

- standard basis vectors in $\mathbb{R}^n$: $(e_1, e_2, \ldots, e_n)$ where $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

  (1 in ith component)

- obviously we have

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

  x_i are called the *coordinates* of x (in the standard basis)

- if $(t_1, t_2, \ldots, t_n)$ is another basis for $\mathbb{R}^n$, we have

$$x = \tilde{x}_1 t_1 + \tilde{x}_2 t_2 + \cdots + \tilde{x}_n t_n$$

  where $\tilde{x}_i$ are the coordinates of x in the basis $(t_1, t_2, \ldots, t_n)$

- then $x = T\tilde{x}$ and $\tilde{x} = T^{-1}x$

Similarity transformation

consider linear transformation $y = Ax$, $A \in \mathbb{R}^{n \times n}$

express y and x in terms of $t_1, t_2 \ldots, t_n$, so $x = T\tilde{x}$ and $y = T\tilde{y}$, then

$$\boxed{\tilde{y} = (T^{-1}AT)\tilde{x}}$$

- $A \longrightarrow T^{-1}AT$ is called *similarity transformation*
- similarity transformation by T expresses linear transformation $y = Ax$ in coordinates $t_1, t_2, \ldots, t_n$